The Physics of Energy

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Random data II

Frequency domain analysis

Fourier transform

The Fourier transform is given by

$$X(f) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i2\pi ft} x(t)\, dt$$

and the inverse transform is given by

$$x(t) = \lim_{F\to\infty} \int_{-F/2}^{F/2} e^{-i2\pi ft} X(f)\, df$$

The Fourier transform is also a random variable

Power spectral density

Average value of the squared magnitude of the Fourier transform

$$S(f) = \langle |X(f)|^2 \rangle = \langle X(f) X^*(f) \rangle$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{i2\pi f t} x(t)\, dt \int_{-T/2}^{T/2} e^{-i2\pi f t} x(t')\, dt'$$

Wiener-Khinchin theorem

$$\int_{-\infty}^{\infty} S(f) e^{-i2\pi f\tau}\, df = \langle x(t)x(t-\tau)\rangle = R_{xx}(\tau)$$

The autocorrelation function of a wide-sense-stationary random process has a spectral decomposition given by the power spectrum of that process

Parseval's theorem

$$\langle x(t)x(t-\tau)\rangle = \int_{-\infty}^{\infty} S(f)e^{-i2\pi f\tau}\,df$$
$$\Rightarrow \langle x^2(t)\rangle = \int_{-\infty}^{\infty} S(f)\,df$$

The average value of the square of the signal (variance if the signal has zero mean) is equal to the integral of the power spectral density

Examples

squared data values. By first subtracting the mean value estimate from all the data values, the mean square value computation yields a variance estimate.

The probability density function $p(x)$ for a stationary record represents the rate of change of probability with data value. The function $p(x)$ is generally estimated by computing the probability that the instantaneous value of the single record will be in a particular narrow amplitude range centered at various data values, and then dividing by the amplitude range. The total area under the probability density function over all data values will be unity, since this merely indicates the certainty of the fact that the data values must fall between $-\infty$ and $+\infty$. The partial area under the probability density function from $-\infty$ to

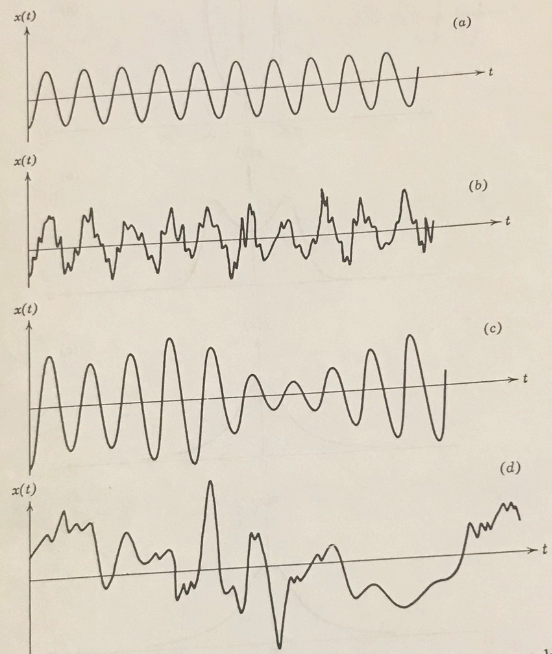


Figure 1.11 Four special time histories. (a) Sine wave. (b) Sine wave plus random noise. (c) Narrow-band random noise. (d) Wide-band random noise.

some given value x represents the probability distribution function, denoted by $P(x)$. The area under the probability density function between any two values x_1 and x_2, given by $P(x_2) - P(x_1)$, defines the probability that any future data values at a randomly selected time will fall within this amplitude interval. Probability density and distribution functions are fully discussed in Chapters 3 and 4.

The autocorrelation function $R_{xx}(\tau)$ for a stationary record is a measure of time-related properties in the data that are separated by fixed time delays. It

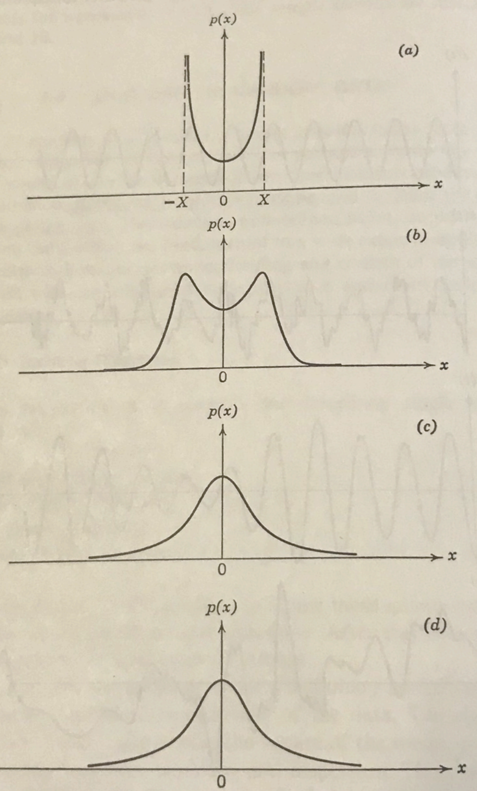


Figure 1.12 Probability density function plots. (*a*) Sine wave. (*b*) Sine wave plus random noise. (*c*) Narrow-band random noise. (*d*) Wide-band random noise.

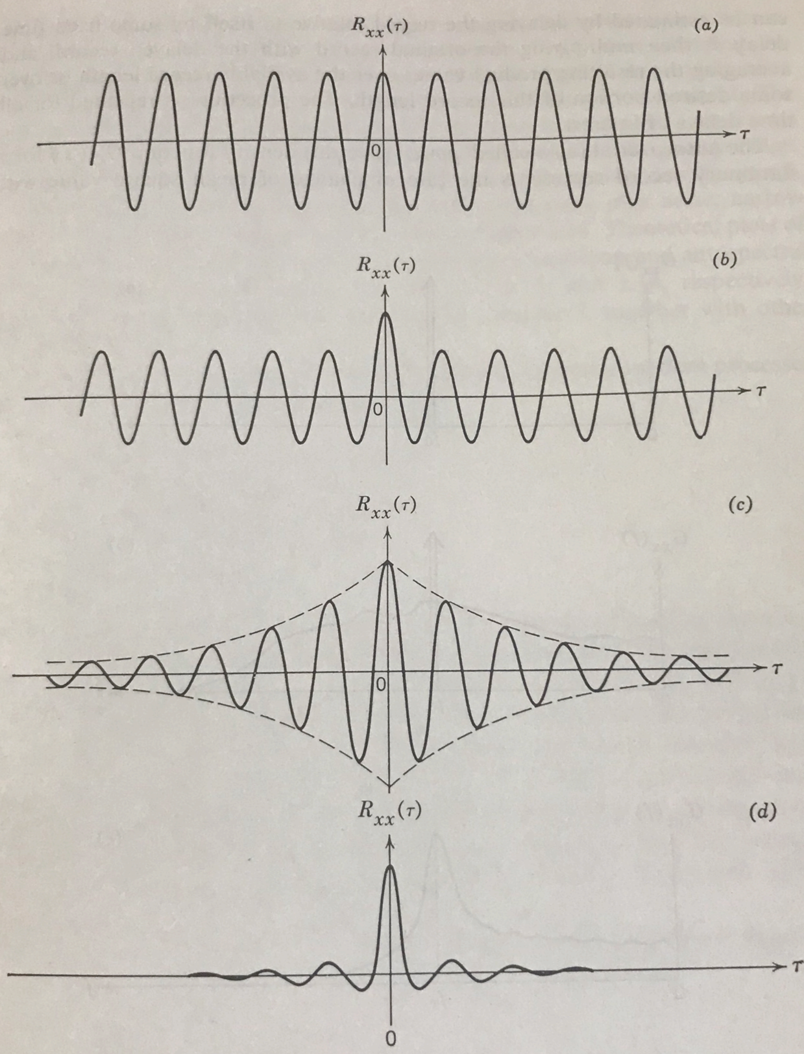


Figure 1.13 Autocorrelation function plots. (*a*) Sine wave. (*b*) Sine wave plus random noise. (*c*) Narrow-band random noise. (*d*) Wide-band random noise.

can be estimated by delaying the record relative to itself by some fixed time delay τ, then multiplying the original record with the delayed record, and averaging the resulting product values over the available record length or over some desired portion of this record length. The procedure is repeated for all time delays of interest.

The autospectral (also called *power* spectral) density function $G_{xx}(f)$ for a stationary record represents the rate of change of mean square value with

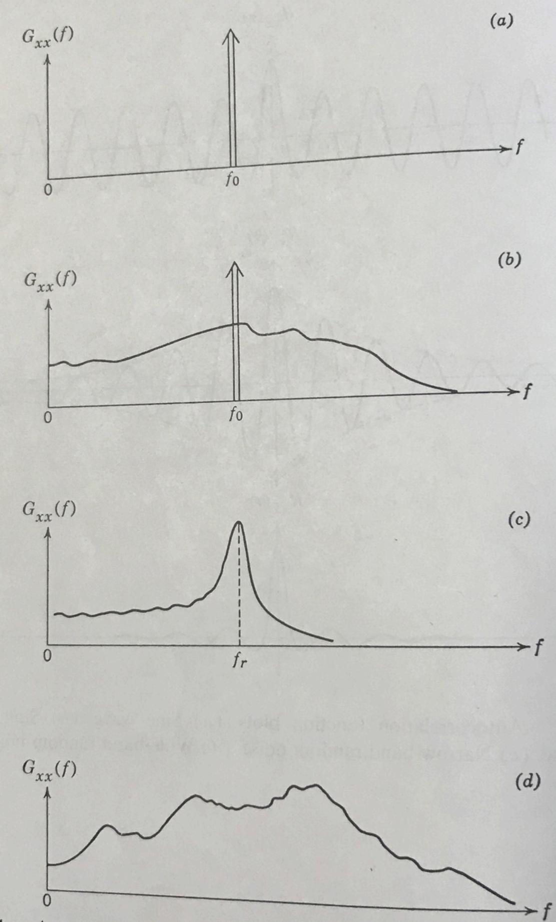
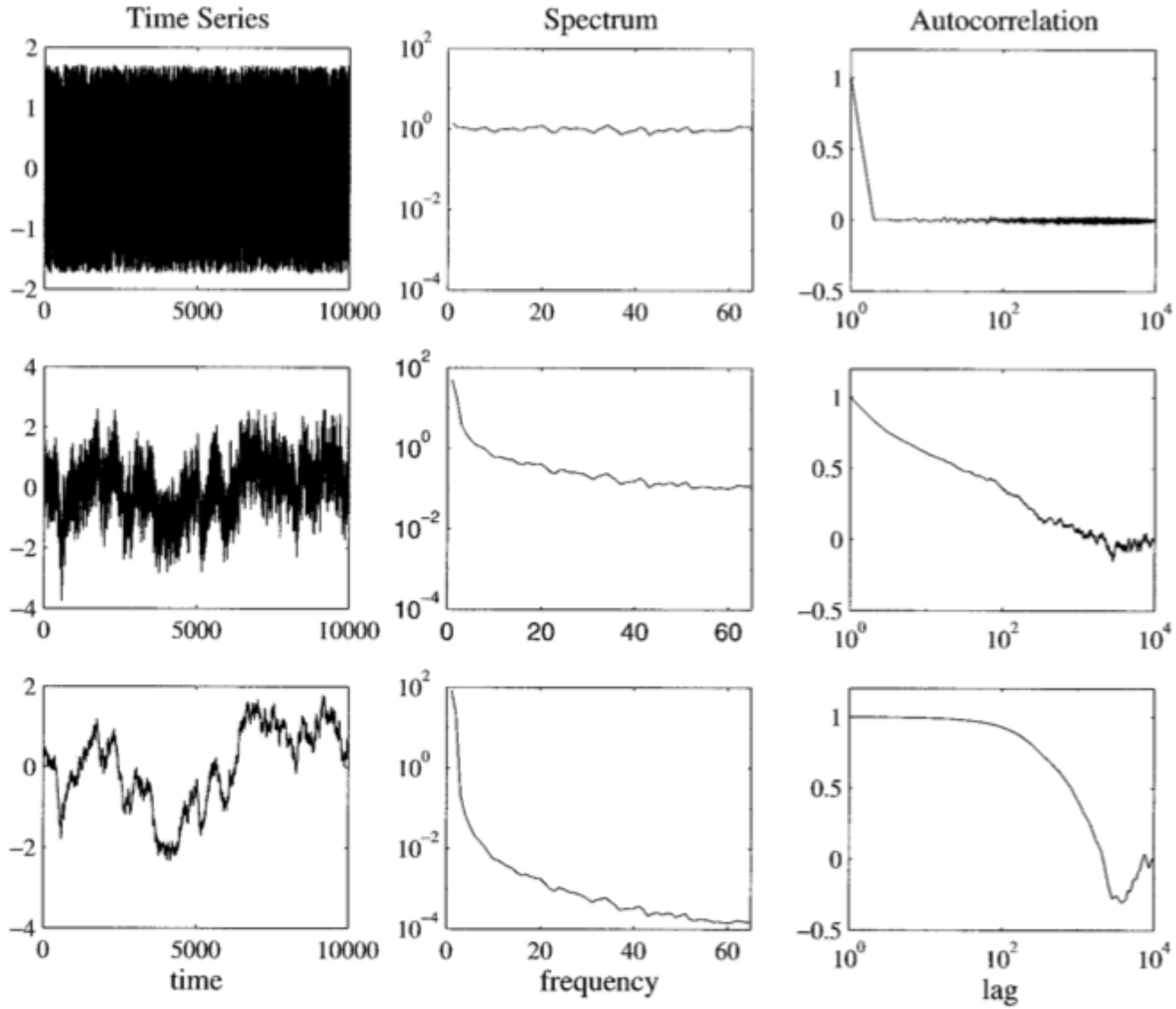


Figure 1.14 Autospectral density function plots. (a) Sine wave. (b) Sine wave plus random noise. (c) Narrow-band random noise. (d) Wide-band random noise.

Examples



White noise

- White noise is a random signal with a **constant** power spectral density
- Sequence of serially **uncorrelated** random variables with **zero mean** and **infinite variance**
- If each sample has a normal distribution with zero mean, the signal is said to be Gaussian white noise

Johnson-Nyquist noise

- Relaxation of thermal fluctuation in a resistor
- Small voltage fluctuation associated with thermal motion of electrons

$$< V_{noise}^2 >= 4kTR\Delta f$$

- Example of fluctuation dissipation relationship

Shot noise

- Generated by discrete arrival
  - electrons in a wire
  - rain on a roof
- Interactions can be ignored
- Arrival independent
  - Poisson process

Shot noise

$$< I >= qN/T$$

$$I(t) = q \sum_{n=1}^{N} \delta(t - t_n)$$

$$I(f) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i2\pi ft} q \sum_{n=1}^{N} \delta(t - t_n)\, dt = q \sum_{n=1}^{N} e^{i2\pi f t_n}$$

$$S_I(f) =< I(f) I^*(f) >= \lim_{T\to\infty} \frac{q^2}{T} \left(\sum_{n=1}^{N} e^{i2\pi f t_n} \sum_{m=1}^{N} e^{-i2\pi f t_m} \right)$$

$$= \lim_{T\to\infty} \frac{q^2 N}{T} = q < I >$$

$$< I_{noise}^2 >= 2q < I > \Delta f$$

1/f and switching noises

- Found in a wide range of transport processes
- The power spectrum diverges at low frequencies inversely proportional to the frequency

$$S(f) \propto f^{-1}$$

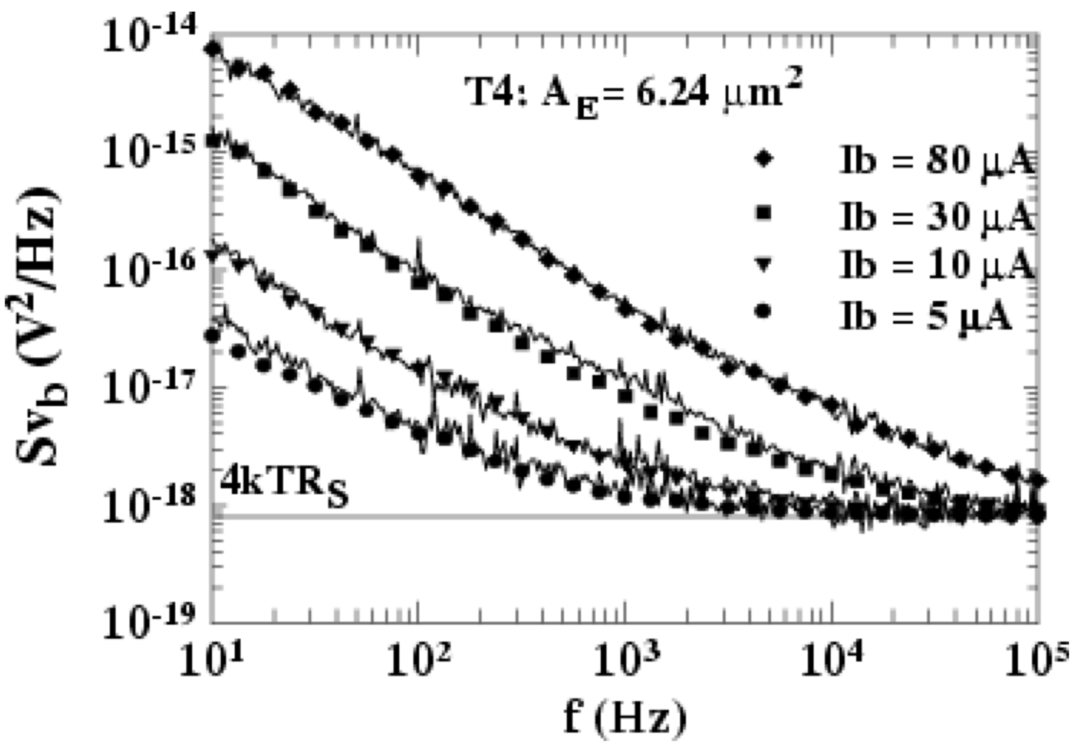
- Scale invariant (look the same at all time scale)

1/*f* and switching noises

- Many type of defects on a conductor
  - lattice vacancies
  - dopant atoms
- Different inequivalent types of sites in the material, which have different energies
- There is a probability for a defect to be excited to an higher-energy state and then relaxed to a lower-energy state

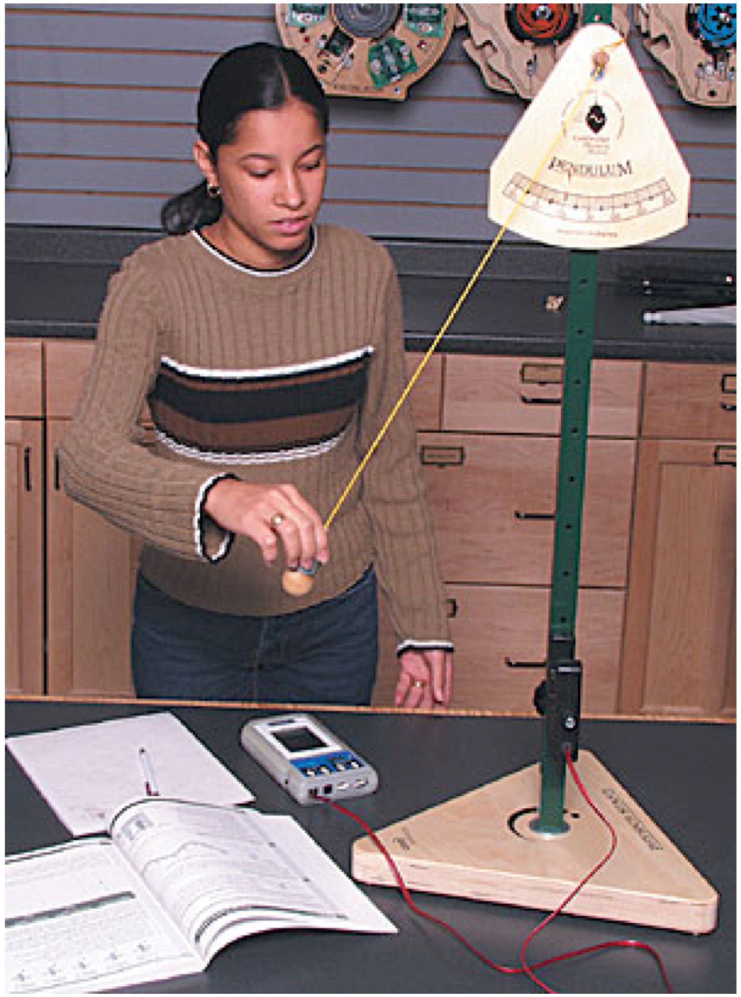
Noise in a transistor

- Flat if no current flowing: Johnson noise
- *1/f* whit current flowing

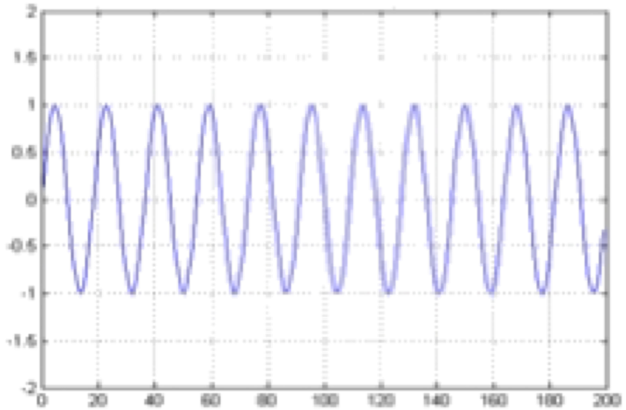


Characterization of polysilicon bipolar transistors by low-frequency noise and correlation noise measurements
Y Mourier, S G-Jarrix, C Delseny, F Pascal, A Pénarier and D Gasquet

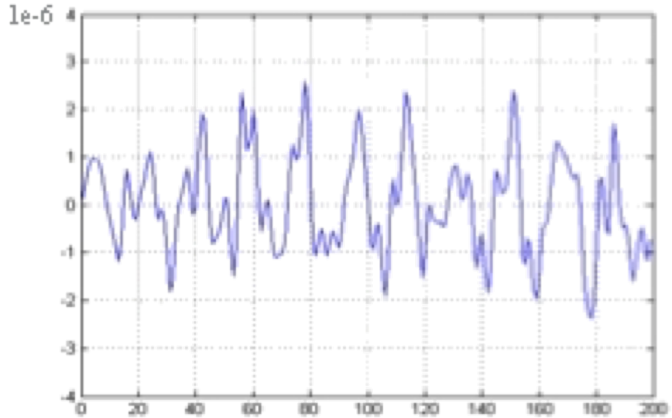
Example: physical system pendulum



Focus on the pendulum angle



If we come back after a while..



Mass m= 1 Kg, Length l = 1 m, **rms motion = 2 10^{-11} m**

To learn more:

Random Data

J. Bendat, A. G. Piersol

Chap. 1 Basic Descriprions and properties